Boolean Algebra

A Boolean Algebra is a mathematical system consisting of a set of elements $B$, two binary operations OR (+) and AND (●), a unary operation NOT ('), an equality sign (=) to indicate equivalence of expressions, and parenthesis to indicate the ordering of the operations, which preserves the following postulates:

P1. The OR operation is closed
   for all $x, y \in B$
   $x + y \in B$

P2. The OR operation has an identity (denoted by 0)
   for all $x \in B$
   $x + 0 = 0 + x = x$

P3. The OR operation is commutative
   for all $x, y \in B$
   $x + y = y + x$

P4. The OR operation distributes over the AND operation
   for all $x, y, z \in B$
   $x + (y \cdot z) = (x + y) \cdot (x + z)$

P5. The AND operation is closed
   for all $x, y \in B$
   $x \cdot y \in B$

P6. The AND operation has an identity (denoted by 1)
   for all $x \in B$
   $x \cdot 1 = 1 \cdot x = x$

P7. The AND operation is commutative
   for all $x, y \in B$
   $x \cdot y = y \cdot x$

P8. The AND operation distributes over the OR operation
   for all $x, y, z \in B$
   $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$

P9. Complement
   for all $x \in B$ there exists an element $x' \in B$, called the complement of $x$, such that
   (a) $x + x' = 1$
   (b) $x \cdot x' = 0$

P10. There exist at least two elements $x, y \in B$ such that $x \neq y$
Theorem 1
*The complement of x is unique*

Proof:

Assume $x_1'$ and $x_2'$ are both complements of x.

Then by P9

$x + x_1' = 1, \ x \cdot x_1' = 0, \ x + x_2' = 1, \ x \cdot x_2' = 0$

Thus, any two elements that are the complement of x are equal.
This implies that $x'$ is unique

Theorem 2-1
**x + 1 = 1**

Proof:

$x + 1 = 1 \cdot (x + 1) \quad 1$ is the identity for AND (P6)

$= (x + x') \cdot (x + 1) \quad$ Complement, $x + x' = 1$ (P9a)

$= x + (x' \cdot 1) \quad$ OR distributes over AND (P4)

$= x + x' \quad 1$ is the identity for AND (P6)

$= 1 \quad$ Complement, $x + x' = 1$ (P9a)
Theorem 3-1

**AND's identity is the complement of OR's identity**

0' = 1

Proof:

\[
0' = 0 + 0' \\
= 1
\]

0 is the identity for OR (P2)
Complement, \( x + x' = 1 \) (P9a)

\[\square\]

Theorem 3-2

**OR's identity is the complement of AND's identity**

1' = 0

Theorem 4-1

Idempotent

\( x + x = x \)

Proof:

\[
x + x = (x + x) \cdot 1 \\
= (x + x) \cdot (x + x') \\
= x + (x \cdot x') \\
= x + 0 \\
= x
\]

1 is the identity for AND (P6)
Complement, \( x + x' = 1 \) (P9a)
OR distributes over AND (P4)
Complement, \( x \cdot x' = 0 \) (P9b)
0 is the identity for OR (P2)

\[\square\]

Theorem 4-2

Idempotent

\( x \cdot x = x \)
Theorem 5

**Involution**

\[(x')' = x\]

Proof:

Let \(x'\) be the complement of \(x\) and \((x')'\) be the complement of \(x'\). Then by P9, \(x + x' = 1, \ xx' = 0, \ x' + (x')' = 1, \) and \(x'(x')' = 0\).

\[
\begin{align*}
(x')' &= (x')' + 0 \\
     &= (x')' + xx' \\
     &= [(x')' + x][(x')' + x'] \\
     &= [x + (x')'][x + (x')'] \\
     &= [x + (x')'] \cdot 1 \\
     &= [x + (x')'][x + x'] \\
     &= x + [(x')' \cdot x'] \\
     &= x + [x' \cdot (x')'] \\
     &= x + 0 \\
     &= x
\end{align*}
\]

0 is the identity for OR (P2)
Substitution, \(xx' = 0\)
OR distributes over AND (P4)
OR is commutative (P3), twice
Substitution, \(x' + (x')' = 1\)
Substitution, \(x + x' = 1\)
OR distributes over AND (P4)
AND is commutative (P7)
Substitution, \(x'(x')' = 0\)
0 is the identity for OR (P2)

\[\square\]

Theorem 6-1

**Absorption**

\[x + xy = x\]

Proof:

\[
\begin{align*}
x + xy &= (x \cdot 1) + xy \\
     &= x(1 + y) \\
     &= x(y + 1) \\
     &= x \cdot 1 \\
     &= x
\end{align*}
\]

1 is the identity for AND (P6)
AND distributes over OR (P8)
OR is commutative (P3)
x + 1 = 1 (Thm 2-1)
1 is the identity for AND (P6)

\[\square\]

Theorem 6-2

**Absorption**

\[x(x + y) = x\]
Theorem 7-1
\[ x + x'y = x + y \]

Proof:
\[
\begin{align*}
x + x'y &= (x + x') (x + y) \\
&= 1 \cdot (x + y) \\
&= x + y
\end{align*}
\]

- OR distributes over AND (P4)
- Complement \( x + x' = 1 \) (P9a)
- 1 is the identity for AND (P6)

\[ \square \]

Theorem 7-2
\[ x(x' + y) = xy \]
Theorem 8-1

**OR is associative**

\[ x + (y + z) = (x + y) + z \]

**Proof:** Let \( A = x + (y + z) \) and \( B = (x + y) + z \)

To Show: \( A = B \)

First,
\[
xA = x \quad [x + (y + z)] \\
= x
\]
Substitution of \( A \)
Absorption \( x(x + y) = x \) (Thm 6-2)

and,
\[
xB = x[(x + y) + z] \\
= x(x + y) + xz \\
= x + xz
\]
AND distributes over OR (P8)
Absorption \( x(x + y) = x \) (Thm 6-2)
Absorption \( x + xy = x \) (Thm 6-1)

Therefore \( xA = xB = x \)

Second,
\[
x'A = x'[x + (y + z)] \\
= x'x + x'(y + z) \\
= xx' + x'(y + z) \\
= 0 + x'(y + z) \\
= x'(y + z)
\]
Complement, \( x \cdot x' = 0 \) (P9b)

and,
\[
x'B = x'[(x + y) + z] \\
= x'(x + y) + x'z \\
= (x'x + x'y) + x'z \\
= (xx' + x'y) + x'z \\
= (0 + x'y) + x'z \\
= x'y + x'z \\
= x'(y + z)
\]
AND distributes over OR (P8)

Therefore \( x'A = x'B = x'(y + z) \)

Finally,
\[
A = A \cdot 1 \\
= A(x + x') \\
= Ax + Ax' \\
= xA + x'A \\
= xB + x'A \\
= xB + x'B \\
= Bx + Bx' \\
= B(x + x') \\
= B \cdot 1 \\
= B
\]
1 is the identity for AND (P6)
Complement, \( x + x' = 1 \) (P9a)
AND distributes over OR (P8)
AND is commutative (P7), twice
Substitution \( xA = xB \)
Substitution \( x'A = x'B \)
AND is commutative (P7), twice
AND distributes over OR (P8)
Complement, \( x + x' = 1 \) (P9a)

Since \( A = x + (y + z) \) and \( B = (x + y) + z \), we have shown that \( x + (y + z) = (x + y) + z \)
**Theorem 8-2**  
*AND is associative*  
\[ x(yz) = (xy)z \]

**Theorem 9-1**  
*DeMorgan’s Law 1*  
\[ (x + y)' = x' \cdot y' \]

**Proof:**  
By Theorem 1 (complements are unique) and Postulate P9 (complement), for every \( x \) in a Boolean algebra there is a unique \( x' \) such that  
\[
 x + x' = 1 \quad \text{and} \quad x \cdot x' = 0
\]

So it is sufficient to show that \( x'y' \) is the complement of \( x + y \). We'll do this by showing that \( (x + y) + (x'y') = 1 \) and \( (x + y) \cdot (x'y') = 0 \)

\[
\begin{align*}
(x + y) + (x'y') &= [(x + y) + x'] [(x + y) + y'] \quad \text{OR distributes over AND (P4)} \\
&= [(y + x) + x'] [(x + y) + y'] \quad \text{OR is commutative (P3)} \\
&= [y + (x + x')] [x + (y + y')] \quad \text{OR is associative (Thm 8-1), twice} \\
&= (y + 1)(x + 1) \quad \text{Complement, } x + x' = 1 \text{ (P9a), twice} \\
&= 1 \cdot 1 \quad x + 1 = 1 \text{ (Thm 2-1), twice} \\
&= 1 \quad \text{Idempotent, } x \cdot x = x \text{ (Thm 4-2)}
\end{align*}
\]

Also,

\[
\begin{align*}
(x + y)(x'y') &= (x'y')(x + y) \quad \text{AND is commutative (P7)} \\
&= [(x'y')x] + [(x'y)y] \quad \text{AND distributes over OR (P8)} \\
&= [(y'x')x] + [(x'y)y] \quad \text{AND is commutative (P7)} \\
&= [y'(x'x)] + [x'(y'y)] \quad \text{AND is associative (Thm 8-2), twice} \\
&= [y' \cdot 0] + [x' \cdot 0] \quad \text{Complement, } x \cdot x' = 0 \text{ (P9b), twice} \\
&= 0 + 0 \quad x \cdot 0 = 0 \text{ (Thm 2-2), twice} \\
&= 0 \quad \text{Idempotent, } x + x = x \text{ (Thm 4-1)}
\end{align*}
\]

\[\square\]

**Theorem 9-2**  
*DeMorgan’s Law 2*  
\[ (xy)' = x' + y' \]
<table>
<thead>
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<th>Property</th>
<th>Description</th>
<th>Ref.</th>
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<td>for all $x, y \in B$, $x + y \in B$</td>
<td>P1</td>
</tr>
<tr>
<td>0 is the identity for OR</td>
<td>$x + 0 = 0 + x = x$</td>
<td>P2</td>
</tr>
<tr>
<td>OR is commutative</td>
<td>$x + y = y + x$</td>
<td>P3</td>
</tr>
<tr>
<td>OR distributes over AND</td>
<td>$x + (y \cdot z) = (x + y) \cdot (x + z)$</td>
<td>P4</td>
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<td>for all $x, y \in B$, $x \cdot y \in B$</td>
<td>P5</td>
</tr>
<tr>
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</tr>
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<td>AND distributes over OR</td>
<td>$x \cdot (y + z) = (x \cdot y) + (x \cdot z)$</td>
<td>P8</td>
</tr>
<tr>
<td>Complement (a)</td>
<td>$x + x' = 1$</td>
<td>P9a</td>
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<td>Complement (b)</td>
<td>$x \cdot x' = 0$</td>
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<td></td>
<td>$x + 1 = 1$</td>
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