Boolean Algebra

A Boolean Algebra is a mathematical system consisting of a set of elements B, two binary operations OR (+) and AND (•), a unary operation NOT ('), an equality sign (=) to indicate equivalence of expressions, and parenthesis to indicate the ordering of the operations, which preserves the following postulates:

P1. The OR operation is closed
for all $x, y \in B$
$x + y \in B$

P2. The OR operation has an identity (denoted by 0)
for all $x \in B$
$x + 0 = 0 + x = x$

P3. The OR operation is commutative
for all $x, y \in B$
$x + y = y + x$

P4. The OR operation distributes over the AND operation
for all $x, y, z \in B$
$x + (y \bullet z) = (x + y) \bullet (x + z)$

P5. The AND operation is closed
for all $x, y \in B$
$x \bullet y \in B$

P6. The AND operation has an identity (denoted by 1)
for all $x \in B$
$x \bullet 1 = 1 \bullet x = x$

P7. The AND operation is commutative
for all $x, y \in B$
$x \bullet y = y \bullet x$

P8. The AND operation distributes over the OR operation
for all $x, y, z \in B$
$x \bullet (y + z) = (x \bullet y) + (x \bullet z)$

P9. Complement
for all $x \in B$ there exists an element $x' \in B$, called the complement of x, such that
(a) $x + x' = 1$
(b) $x \bullet x' = 0$

P10. There exist at least two elements $x, y \in B$ such that $x \neq y$

Theorem 1
The complement of x is unique

Proof :

Assume x_1' and x_2' are both complements of x.

Then by P9

$$x + x_1' = 1, \quad x \bullet x_1' = 0, \quad x + x_2' = 1, \quad x \bullet x_2' = 0$$

$x_1' = x_1' \bullet 1$	1 is the identity for AND (P6)
$= x_1' \bullet (x + x_2')$	substitution, $x + x_2' = 1$
$= (x_1' \bullet x) + (x_1' \bullet x_2')$	AND distributes over OR (P8)
$= (x \bullet x_1') + (x_1' \bullet x_2')$	AND is commutative (P7)
$= 0 + (x_1' \bullet x_2')$	substitution, $x \bullet x_1' = 0$
$= (x \bullet x_2') + (x_1' \bullet x_2')$	substitution, $x \bullet x_2' = 0$
$= (x_2' \bullet x) + (x_2' \bullet x_1')$	AND is commutative (P7), twice
$= x_2' \bullet (x + x_1')$	AND distributes over OR (P8)
$= x_2' \bullet 1$	substitution, $x + x_1' = 1$
$= x_2'$	1 is the identity for AND (P6)

Thus, any two elements that are the complement of x are equal.
This implies that x' is unique

Theorem 2-1
x + 1 = 1

Proof:

$x + 1 = 1 \bullet (x + 1)$	1 is the identity for AND (P6)
$= (x + x') \bullet (x + 1)$	Complement, $x + x' = 1$ (P9a)
$= x + (x' \bullet 1)$	OR distributes over AND (P4)
$= x + x'$	1 is the identity for AND (P6)
$= 1$	Complement, $x + x' = 1$ (P9a)

□

Theorem 2-2
x • 0 = 0

Theorem 3-1
AND's identity is the complement of OR's identity
0' = 1

Proof:

$0' = 0 + 0'$	0 is the identity for OR (P2)
$= 1$	Complement, $x + x' = 1$ (P9a)

□

Theorem 3-2
OR's identity is the complement of AND's identity
1' = 0

Theorem 4-1
Idempotent
x + x = x

Proof:

$x + x = (x + x) \bullet 1$	1 is the identity for AND (P6)
$= (x + x) \bullet (x + x')$	Complement, $x + x' = 1$ (P9a)
$= x + (x \bullet x')$	OR distributes over AND (P4)
$= x + 0$	Complement, $x \bullet x' = 0$ (P9b)
$= x$	0 is the identity for OR (P2)

□

Theorem 4-2
Idempotent
x • x = x

Theorem 5

Involution

(x')' = x

Proof:

Let x' be the complement of x and (x')' be the complement of x'.
Then by P9, $x + x' = 1$, $xx' = 0$, $x' + (x')' = 1$, and $x'(x')' = 0$

$(x')' = (x')' + 0$	0 is the identity for OR (P2)
$= (x')' + xx'$	Substitution, $xx' = 0$
$= [(x')' + x][(x')' + x']$	OR distributes over AND (P4)
$= [x + (x')'][x' + (x')']$	OR is commutative (P3), twice
$= [x + (x')'] \bullet 1$	Substitution, $x' + (x')' = 1$
$= [x + (x')'][x + x']$	Substitution, $x + x' = 1$
$= x + [(x')' \bullet x']$	OR distributes over AND (P4)
$= x + [x' \bullet (x')']$	AND is commutative (P7)
$= x + 0$	Substitution, $x'(x')' = 0$
$= x$	0 is the identity for OR (P2)

□

Theorem 6-1

Absorption

x + xy = x

Proof:

$x + xy = (x \bullet 1) + xy$	1 is the identity for AND (P6)
$= x(1 + y)$	AND distributes over OR (P8)
$= x(y + 1)$	OR is commutative (P3)
$= x \bullet 1$	$x + 1 = 1$ (Thm 2-1)
$= x$	1 is the identity for AND (P6)

□

Theorem 6-2

Absorption

x(x + y) = x

Theorem 7-1
$x + x'y = x + y$

Proof:

$x + x'y = (x + x')(x + y)$	OR distributes over AND (P4)
$= 1 \bullet (x + y)$	Complement $x + x' = 1$ (P9a)
$= x + y$	1 is the identity for AND (P6)

□

Theorem 7-2
$x(x' + y) = xy$

Theorem 8-1
OR is associative
$x + (y + z) = (x + y) + z$

Proof: Let $A = x + (y + z)$ and $B = (x + y) + z$
To Show: $A = B$

First,

$xA = x[x + (y + z)]$	Substitution of A
$= x$	Absorption $x(x + y) = x$ (Thm 6-2)

and,

$xB = x[(x + y) + z]$	Substitution of B
$= x(x + y) + xz$	AND distributes over OR (P8)
$= x + xz$	Absorption $x(x + y) = x$ (Thm 6-2)
$= x$	Absorption $x + xy = x$ (Thm 6-1)

Therefore $xA = xB = x$

Second,

$x'A = x'[x + (y + z)]$	Substitution of A
$= x'x + x'(y + z)$	AND distributes over OR (P8)
$= xx' + x'(y + z)$	AND is commutative (P7)
$= 0 + x'(y + z)$	Complement, $x \bullet x' = 0$ (P9b)
$= x'(y + z)$	0 is the identity for OR (P2)

and.

$x'B = x'[(x + y) + z]$	Substitution of B
$= x'(x + y) + x'z$	AND distributes over OR (P8)
$= (x'x + x'y) + x'z$	AND distributes over OR (P8)
$= (xx' + x'y) + x'z$	AND is commutative (P7)
$= (0 + x'y) + x'z$	Complement, , $x \bullet x' = 0$ (P9b)
$= x'y + x'z$	0 is the identity for OR (P2)
$= x'(y + z)$	AND distributes over OR (P8)

Therefore $x'A = x'B = x'(y + z)$

Finally,

$A = A \bullet 1$	1 is the identity for AND (P6)
$= A(x + x')$	Complement, $x + x' = 1$ (P9a)
$= Ax + Ax'$	AND distributes over OR (P8)
$= xA + x'A$	AND is commutative (P7), twice
$= xB + x'A$	Substitution $xA = xB$
$= xB + x'B$	Substitution $x'A = x'B$
$= Bx + Bx'$	AND is commutative (P7), twice
$= B(x + x')$	AND distributes over OR (P8)
$= B \bullet 1$	Complement, $x + x' = 1$ (P9a)
$= B$	1 is the identity for AND (P6)

Since $A = x + (y + z)$ and $B = (x + y) + z$, we have shown that $x + (y + z) = (x + y) + z$
□

Theorem 8-2
AND is associative
$x(yz) = (xy)z$

Theorem 9-1
DeMorgan's Law 1
$(x + y)' = x' y'$

Proof:

By Theorem 1 (complements are unique) and Postulate P9 (complement), for every x in a Boolean algebra there is a unique x' such that

$x + x' = 1$ and $x \bullet x' = 0$

So it is sufficient to show that x'y' is the complement of x + y. We'll do this by showing that $(x + y) + (x'y') = 1$ and $(x + y) \bullet (x'y') = 0$

$(x + y) + (x'y')$	$= [(x + y) + x'] [(x + y) + y']$	OR distributes over AND (P4)
	$= [(y + x) + x'] [(x + y) + y']$	OR is commutative (P3)
	$= [y + (x + x')] [x + (y + y')]$	OR is associative (Thm 8-1), twice
	$= (y + 1)(x + 1)$	Complement, $x + x' = 1$ (P9a), twice
	$= 1 \bullet 1$	$x + 1 = 1$ (Thm 2-1), twice
	$= 1$	Idempotent, $x \bullet x = x$ (Thm 4-2)

Also,

$(x + y)(x'y')$	$= (x'y')(x + y)$	AND is commutative (P7)
	$= [(x'y')x] + [(x'y')y]$	AND distributes over OR (P8)
	$= [(y'x')x] + [(x'y')y]$	AND is commutative (P7)
	$= [y'(x'x)] + [x'(y'y)]$	AND is associative (Thm 8-2), twice
	$= [y'(xx')] + [x'(yy')]$	AND is commutative (P7), twice
	$= [y' \bullet 0] + [x' \bullet 0]$	Complement, $x \bullet x' = 0$ (P9b), twice
	$= 0 + 0$	$x \bullet 0 = 0$ (Thm 2-2), twice
	$= 0$	Idempotent, $x + x = x$ (Thm 4-1)

□

Theorem 9-2
DeMorgan's Law 2
$(xy)' = x' + y'$

Summary

OR is closed	for all $x, y \in B, x + y \in B$	P1
0 is the identity for OR	$x + 0 = 0 + x = x$	P2
OR is commutative	$x + y = y + x$	P3
OR distributes over AND	$x + (y \bullet z) = (x + y) \bullet (x + z)$	P4
AND is closed	for all $x, y \in B, x \bullet y \in B$	P5
1 is the identity for AND	$x \bullet 1 = 1 \bullet x = x$	P6
AND is commutative	$x \bullet y = y \bullet x$	P7
AND distributes over OR	$x \bullet (y + z) = (x \bullet y) + (x \bullet z)$	P8
Complement (a)	$x + x' = 1$	P9a
Complement (b)	$x \bullet x' = 0$	P9b
Complements are unique		Thm 1
	$x + 1 = 1$	Thm 2-1
	$x \bullet 0 = 0$	Thm 2-2
	$0' = 1$	Thm 3-1
	$1' = 0$	Thm 3-2
Idempotent	$x + x = x$	Thm 4-1
Idempotent	$x \bullet x = x$	Thm 4-2
Involution	$(x')' = x$	Thm 5
Absorption	$x + xy = x$	Thm 6-1
Absorption	$x (x + y) = x$	Thm 6-2
	$x + x'y = x + y$	Thm 7-1
	$x (x' + y) = xy$	Thm 7-2
OR is associative	$x + (y + z) = (x + y) + z$	Thm 8-1
AND is associative	$x(yz) = (xy)z$	Thm 8-2
DeMorgan's Law 1	$(x + y)' = x' y'$	Thm 9-1
DeMorgan's Law 2	$(xy)' = x' + y'$	Thm 9-2